PMM U.S.S.R.,Vol.47,No.2,pp.162-170,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.375

ASYMPTOTICS NEAR THE TIP OF A CRACK OF THE STATE OF STRESS AND STRAIN OF INHOMOGENEOUSLY AGING BODIES*

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The asymptotic behavior is studied of the solution of the creep theory problem of inhomogeneously aging bodies in the neighborhood of a crack tip. Asymptotic representations of the stresses and displacements are obtained. It turns out that these representations for the stresses agree with the correspondong representations in classical elasticity theory, while they differ for the displacements by additional terms. The formulas obtained for the displacements extend the results in /1 - 3/ in which the agreement between the asymptotics for the stresses in creep and elasticity problems is confirmed for a homogeneous material.

1. Let Ω be a plane, connected domain with smooth boundaries containing a slit $\Gamma = \{\mathbf{x} \mid x_2 = 0,\ a \leqslant x_1 \leqslant b\}$. The equations of plane strain in Cartesian coordinates are obtained from the equations of the three-dimensional problem /4/ by substituting the conditions (everywhere henceforth $i, j = 1, 2$)

$$u_i(t, x_1, x_2, x_3) = u_i(t, x_1, x_2),\ u_3 = 0;\ \varepsilon_{ij} = \varepsilon_{ij}(\mathbf{u}) \equiv (u_{i,j} + u_{j,i})/2 \tag{1.1}$$

$$\sigma_{ij,j}(t, \mathbf{x}) + f_i(t, \mathbf{x}) = 0,\ \mathbf{x} \in \Omega \setminus \Gamma \tag{1.2}$$

$$\frac{s_{ij}(t, \mathbf{x})}{2G[t + \varkappa(\mathbf{x}), \mathbf{x}]} = e_{ij}(t, \mathbf{x}) - \int_0^t R_1[t + \varkappa(\mathbf{x}), \tau + \varkappa(\mathbf{x}), \mathbf{x}]\, e_{ij}(\tau, \mathbf{x})\, d\tau \tag{1.3}$$

$$\frac{\sigma(t, \mathbf{x})}{E_*[t + \varkappa(\mathbf{x}), \mathbf{x}]} = e(t, \mathbf{x}) - \int_0^t R_2[t + \varkappa(\mathbf{x}), \tau + \varkappa(\mathbf{x}), \mathbf{x}]\, e(\tau, \mathbf{x})\, d\tau$$

$$e = (e_{11} + e_{22})/3,\ \varepsilon_{ij} = \delta_{ij} e + e_{ij},\ \sigma_{ij} = \delta_{ij}\sigma + s_{ij} \tag{1.4}$$

$$u_i = 0,\ \mathbf{x} \in S_u;\ \sigma_{ij} n_j = P_i(t, \mathbf{x}),\ \mathbf{x} \in S_p;\ S_n \cup S_p = \partial\Omega \tag{1.5}$$

$$\sigma_{12} = g_1^{\pm}(t, \mathbf{x}),\ \sigma_{22} = g_2^{\pm}(t, \mathbf{x}),\ \mathbf{x} \in \Gamma^{\pm}$$

Here u_i, σ_{ij}, ε_{ij} are the Cartesian components of the displacement, stress, and strain, respectively, s_{ij}, e_{ij} are the components of the stress and strain deviators, σ, e are their spherical parts, $E_*(t, \mathbf{x})$, $R_2(t, \tau, \mathbf{x})$ are the volume expansion modulus and the relaxation kernel under multilateral tension (compression), $G(t, \mathbf{x})$, $R_1(t, \tau, \mathbf{x})$ are the shear modulus and the relaxation kernel under shear, $\varkappa(\mathbf{x})$ is a function of inhomogeneous aging characterizing the law of variation of material growth, and f_i, P_i, $g_i^{\pm}$ are the volume and surface loads.

We shall consider the solution in an arbitrary time segment $[0, T]$. Let us formulate the constraints under which the solution of the creep problem exists. For $\forall t$ let the loads f_i, P_i, $g_i^{\pm}$ be square summable and piecewise continuous in t (i.e., instantaneous changes in the load are allowed at separate times) as a mapping of the segment $[0, T]$ in the space L_2, let the moduli E_*, G be continous in t, piecewise continuous in $\mathbf{x}$ and satisfy the estimates

$$E_1 \leqslant E_* \leqslant E_2,\ G_1 \leqslant G \leqslant G_2,\ E_1,\ E_2,\ G_1,\ G_2 = \text{const} > 0$$

The relaxation kernels R_i are representable in the form

$$R_i(t, \tau, \mathbf{x}) = p_i(t, \tau, \mathbf{x})(t - \tau)^{-\alpha} + q_i(t, \tau, \mathbf{x}),\ \alpha < 1 \tag{1.6}$$

where the p_i, q_i are bounded and continuous in t, τ, piecewise continuous in $\mathbf{x}$, and the function $\varkappa(\mathbf{x})$ is bounded and piecewise continuous. It is known /5/ that under these constraints there exists a unique generalized solution of the problem (1.1)-(1.5). From the manner of

*Prikl.Matem.Mekhan.,Vol.47,No.2,pp.200-208,1983

the proof of this assertion in /5/ and the known results /6/ on the smoothness of the solution of elliptic systems, there follows the local smoothness of the solution of the creep problem in the coordinates $\mathbf{x}$ in regularity subdomains of the right side and the rheological characteristics.

2. For definiteness, we will study the solution in the neighborhood U of the right tip of the crack $\mathbf{x}_B$. It is assumed that the functions E_*, G, p_i, q_i are smooth in the set of arguments t, τ, $\mathbf{x}$ for $t, \tau \in [0, T]$, $\mathbf{x} \in U$. The functions f_i are smooth in $\mathbf{x}$ for $t \in [0, T]$, $\mathbf{x} \in U$, and the functions $g_i^\pm$ are smooth in $\mathbf{x}$ for $t \in [0, T]$, $\mathbf{x} \in U \cap \Gamma^\pm$, where

$$g(t, \mathbf{x}_B) \equiv g_i^+(t, \mathbf{x}_B) = g_i^-(t, \mathbf{x}_B), \quad t \in [0, \ T] \tag{2.1}$$

(Here smoothness is understood to be the presence of a sufficient number of continuous derivatives, where it is assumed that the derivatives of the functions f_i, g_i with respect to the space coordinates are piecewise continuous in t as a mapping of the segment $[0, T]$ in the space of continuous functions). Then the solution is smooth in the domain $U \setminus D_d$, where D_d is a circle of radius d with center at $\mathbf{x}_B$, and d is arbitrary. Let us select d such that $D_{2d} \subset U$, and let us introduce a smooth cutoff function $\chi(\mathbf{x})$ such that $\chi = 1$ for $\mathbf{x} \in D_{d/2}$ and $\chi = 0$ for $\mathbf{x} \not\in D_d$. Let (r, θ) be polar coordinates with origin at $\mathbf{x}_B$ and polar axis directed along the segment Γ such that the equalities $\theta = 0$ and $\theta = 2\pi$ hold, respectively, on $\Gamma^\mp$

Let us introduce a notation for the rheological characteristics "frozen" at the crack tip $\mathbf{x}_B$

$$G^\circ(t) \equiv G[t + \varkappa(\mathbf{x}_B), \ \mathbf{x}_B], \quad E_*^\circ(t) = E_*[t + \varkappa(\mathbf{x}_B), \ \mathbf{x}_B]$$
$$\nu \equiv (E_* - 2G)/(2G + 2E_*), \qquad k \equiv 3 - 4\nu$$
$$\nu^\circ(t) \equiv \nu[t + \varkappa(\mathbf{x}_B), \ \mathbf{x}_B], \quad k^\circ(t) \equiv k[t + \varkappa(\mathbf{x}_B), \ \mathbf{x}_B]$$
$$R_i^\circ(t, \tau) \equiv R_i[t + \varkappa(\mathbf{x}_B), \ \tau + \varkappa(\mathbf{x}_B), \ \mathbf{x}_B]$$

Theorem. Under the assumptions made, asymptotic representations of the solution of the creep problem are valid ($\mathbf{l}$ is the displacement of the body as a rigid whole)

$$\mathbf{u}(t, r, \theta) = \begin{Vmatrix} u_r(t, r, \theta) \\ u_\theta(t, r, \theta) \end{Vmatrix} = r^{1/2}[C_1(t)\boldsymbol{\psi}^1(t, \theta) + C_2(t)\boldsymbol{\psi}^2(t, \theta) + A_1(t)\boldsymbol{\xi}^1(\theta) + A_2(t)\boldsymbol{\xi}^2(\theta)]\chi(r) + \mathbf{o}(r) + \mathbf{l}(t, \mathbf{x}) \tag{2.2}$$

$$\boldsymbol{\psi}^1(t, \theta) = \begin{Vmatrix} \psi_r^1(t, \theta) \\ \psi_\theta^1(t, \theta) \end{Vmatrix} \equiv \begin{Vmatrix} [2k^\circ(t) - 1]\sin\theta/2 + \sin 3\theta/2 \\ [2k^\circ(t) + 1]\cos\theta/2 + \cos 3\theta/2 \end{Vmatrix}$$

$$\boldsymbol{\psi}^2(t, \theta) = \begin{Vmatrix} \psi_r^2(t, \theta) \\ \psi_\theta^2(t, \theta) \end{Vmatrix} \equiv \begin{Vmatrix} [2k^\circ(t) - 1]\cos\theta/2 + 3\cos 3\theta/2 \\ -[2k^\circ(t) + 1]\sin\theta/2 - 3\sin 3\theta/2 \end{Vmatrix}$$

$$\boldsymbol{\xi}^1(\theta) = \begin{Vmatrix} \xi_r^1(\theta) \\ \xi_\theta^1(\theta) \end{Vmatrix} \equiv \begin{Vmatrix} \sin\theta/2 \\ \cos\theta/2 \end{Vmatrix}, \quad \boldsymbol{\xi}^2(\theta) = \begin{Vmatrix} \xi_r^2(\theta) \\ \xi_\theta^2(\theta) \end{Vmatrix} \equiv \begin{Vmatrix} \cos\theta/2 \\ -\sin\theta/2 \end{Vmatrix}$$

$$\mathbf{l}(t, \mathbf{x}) = \begin{Vmatrix} l_1(t, \mathbf{x}) \\ l_2(t, \mathbf{x}) \end{Vmatrix} = \begin{Vmatrix} a_1(t) + b(t)x_2 \\ a_2(t) - b(t)x_1 \end{Vmatrix}$$

$$\begin{aligned} \sigma_{rr}(t, r, \theta) &= 2G^\circ(t)r^{-1/2}[B_1(t)(^5/_2\sin\theta/2 + {}^1/_2\sin 3\theta/2) + B_2(t)(^5/_2\cos\theta/2 + {}^3/_2\cos 3\theta/2)]\chi(r) + o(1) \\ \sigma_{\theta\theta}(t, r, \theta) &= 2G^\circ(t)r^{-1/2}[B_1(t)(^3/_2\sin\theta/2 - {}^1/_2\sin 3\theta/2) + B_2(t)(^3/_2\cos\theta/2 - {}^3/_2\cos 3\theta/2)]\chi(r) + o(1) \\ \sigma_{r\theta}(t, r, \theta) &= 2G^\circ(t)r^{-1/2}[B_1(t)(-\cos\theta/2 + \cos 3\theta/2) + B_2(t)(\sin\theta/2 - 3\sin 3\theta/2)]\chi(r) + o(1) \end{aligned} \tag{2.3}$$

The coefficients $A_i(t)$ in (2.2) are determined in terms of $C_i(t)$ from the solution of a Volterra integral equation of the second kind

$$A_i(t) + \frac{2[1 + \nu^\circ(t)]}{3}\int_0^t Z(t, \tau)A_i(\tau)\,d\tau - \int_0^t R_1^\circ(t, \tau)A_i(\tau)\,d\tau + \frac{8[1 + \nu^\circ(t)]}{3}\int_0^t Z(t, \tau)[1 - 2\nu^\circ(\tau)]C_i(\tau)\,d\tau - \tag{2.4}$$

$$8\int_0^t R_1^\circ(t,\tau)\,[\nu^\circ(t)-\nu^\circ(\tau)]\,C_i(\tau)\,d\tau=0$$
$$Z(t,\tau)=R_1^\circ(t,\tau)-R_2^\circ(t,\tau)$$

The coefficients $B_i(t)$ in (2.3) are expressed in terms of $C_i(t)$ by the formulas

$$B_i(t)=C_i(t)-\int_0^t R_1^\circ(t,\tau)\,C_i(\tau)\,d\tau$$

Formulas (2.2) allow term-by-term differentiation.

Remark 1°. The asymptotic representations (2.3) for the stresses agree with the corresponding representations of the elastic problem /7,8/.

2°. If the kernels of volume and shear relaxation agree at the crack tip, and the Poisson's ratio is $\nu^\circ=\mathrm{const}$, then $A_i\equiv 0$ follows from (2.4), and the representations (2.2) agree with the corresponding representations of the elastic problem. In the case of a homogeneous domain this result follows directly from theorems of N.Kh. Arutiunian /9 - 11/.

Proof. Let B be a Banach space, and let $L^\infty(0,t;B)$ denote the space of mappings Z of the segment $[0,t]$ into B allotted by the norm

$$|Z,B,t|=\operatorname*{esssup}_{0\leqslant\tau\leqslant t}\|Z(\tau),B\|$$

Then from the assumptions made above, the imbeddings

$$f_i\in L^\infty[0,T;\ L_2(\Omega)],\ P_i\in L^\infty[0,T;\ L_2(S_p)],\ g_i^\pm\in L^\infty[0,T;\ L_2(\Gamma^\pm)]$$

follow.

Let $v_i\equiv u_i\chi$, then the v_i agree with u_i in the domain $D_{d/2}$, equal zero outside D_d, and are smooth outside the domain $D_{d/2}$. Let $\sigma_{ij}(\mathbf{v})$ denote the stress of the creep problem that corresponds to the displacements $\mathbf{v}$, i.e., those obtained by substituting $\mathbf{v}$ into (1.1) and (1.3), and by assumption we set $F_i\equiv-\sigma_{ij,j}(\mathbf{v})$ and $Q_i^\pm=\sigma_{i2}^\pm(\mathbf{v})$. Evidently the equalities $\sigma_{ij}(\mathbf{v})=\sigma_{ij}(\mathbf{u})$, $F_i=f_i$ and $Q_i^\pm=g_i^\pm$ are valid in the circle $D_{d/2}$ and the functions F_i, and $Q_i^\pm$ are smooth outside the circle $D_{d/2}$.

The boundary ∂D_{2d} of the circle D_{2d} makes right angles with $\Gamma^\pm$, we "round off" these angles by inscribing small arcs, smoothly tangent to $\Gamma^\pm$ and ∂D_{2d}. The contour γ obtained consists of segments of the crack edges, the round-off arc, and parts of the boundary ∂D_{2d}. The domain bounded by the contour γ is denoted by ω.

The functions $\mathbf{v}$, $\varepsilon_{ij}(\mathbf{v})$, $\sigma_{ij}(\mathbf{v})$ are a solution of the creep problem in the domain ω for the volume loads $\mathbf{F}=(F_1,F_2)$ and the surface loads $\mathbf{Q}=(Q_1,Q_2)$, where $Q_i=\mp Q_i^\pm$ on $\gamma\cap\Gamma^\pm$, and $Q_i=0$ on $\gamma\setminus(\Gamma^+\cup\Gamma^-)$.

Let $H(\Omega)$ denote the S.L. Sobolev space, and $\|v;H^k(\Omega)\|$ the norm of the function v in $H^k(\Omega)$. Following /12,13/, let $V_\beta^k(\omega)$ denote the space of functions v in ω with the norm

$$\|v;\ V_\beta^k(\omega)\|=\sum_{j=0}^k\|r^{\beta+j-k}v;\ H^j(\omega)\|,\quad k=0,1,\ldots$$

where β is a real number. We let $V_\beta^{k-1/2}(\gamma)$ denote the space of traces of functions from $V_\beta^k(\omega)$ on γ.

For brevity, we write the equation of the creep problem in displacements in operator form

$$\mathbf{Lv}-\mathbf{L}^R\mathbf{v}+\mathbf{F}=\mathbf{0},\quad \mathbf{x}\in\omega;\quad \mathbf{Bv}-\mathbf{B}^R\mathbf{v}=\mathbf{Q},\quad \mathbf{x}\in\gamma \tag{2.5}$$

Here $\mathbf{L}$ is the operator of the equilibrium equations of the instantaneous elastic problem, $\mathbf{L}^R$ is an operator containing all the integral components, $\mathbf{Bv}$ is the instantaneous elastic surface stress vector, $\mathbf{B}^R\mathbf{v}$ are the integral components of the surface stress vector.

As usual, we seek the solution of the problem (2.5) in the form of series

$$\mathbf{v}(t,r,\theta)=\sum_{n=1}^\infty\mathbf{v}^n(t,r,\theta) \tag{2.6}$$

whose coefficients satisfy boundary value problems with the parameter $t \in [0, T]$

$$\mathbf{L}\mathbf{v}^1 + \mathbf{F} = 0, \quad \mathbf{x} \in \omega; \quad \mathbf{B}\mathbf{v}^1 = \mathbf{Q}, \quad \mathbf{x} \in \gamma \tag{2.7}$$

$$\mathbf{L}\mathbf{v}^n + \mathbf{F}^{n-1} = 0, \quad \mathbf{x} \in \omega; \quad \mathbf{B}\mathbf{v}^n = \mathbf{Q}^{n-1}, \quad \mathbf{x} \in \gamma, \quad n = 2, 3, \ldots \tag{2.8}$$

$$\mathbf{F}^{n-1} \equiv -\mathbf{L}^R\mathbf{v}^{n-1}, \quad \mathbf{Q}^{n-1} = \mathbf{B}^R\mathbf{v}^{n-1}$$

It is known /5/ that under the assumptions made the series (2.6) converges to a generalized solution of the problem (2.5) in $L^\infty[0, T; H^1(\omega)]$

The first term of the series is the solution of the instantaneously elastic problem, and the next terms are determined from the solution of the elasticity problem equations with right side dependent on the preceding approximation. It can be shown that the equilibrium conditions assuring solvability of the elasticity problem for each t are satisfied at each step.

Let us examine the problem (2.7) for any $t \in [0, T]$ under the assumption that $\mathbf{F}, \mathbf{Q}$ are arbitrary effects satisfying the equilibrium conditions, that are representable in the form

$$\mathbf{F} = r^{-3/2} m\, (A_1 \boldsymbol{\zeta}^1 + A_2 \boldsymbol{\zeta}^2)\chi + \mathbf{F}^* \tag{2.9}$$

$$\boldsymbol{\zeta}^1 = \begin{Vmatrix} \zeta_r^1 \\ \zeta_\theta^1 \end{Vmatrix} = \begin{Vmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{Vmatrix}, \quad \boldsymbol{\zeta}^2 = \begin{Vmatrix} \zeta_r^2 \\ \zeta_\theta^2 \end{Vmatrix} = \begin{Vmatrix} \cos\theta/2 \\ \sin\theta/2 \end{Vmatrix}$$

$$\mathbf{Q} = \begin{Vmatrix} Q_r \\ Q_\theta \end{Vmatrix} = \begin{Vmatrix} Q_r^* - \alpha_1\chi\cos\theta/2 \\ Q_\theta^* - (2mA_2 r^{-1/2} + \alpha_2\cos\theta/2)\chi \end{Vmatrix}$$

$$\mathbf{F}^* \in \overset{\circ}{V}_{-\delta}(\omega), \quad \mathbf{Q}^* \in V_{-\delta}^{1/2}(\gamma), \quad \delta > 0, \quad m = m(t) \equiv \frac{G^\circ(t)}{2[1 - 2\nu^\circ(t)]}$$

($A_1, A_2, \alpha_1, \alpha_2$ are constants). The solution is determined to the accuracy of the body displacement as a rigid whole, hence, we impose the additional constraints

$$\int_\omega \mathbf{v}\, d\omega = 0, \quad \int_\omega \operatorname{rot} \mathbf{v}\, d\omega = 0 \tag{2.10}$$

The following result holds, which follows from /12,13/ (a sufficiently detailed interpretation of the results of /12,13/ on general elliptic boundary value problems in domains with conic points as applied to elasticity theory is contained in /14,15/). The problem (2.7), (2.10) under the conditions (2.9) is solvable uniquely, and the solution is representable in the form

$$\mathbf{v} = r^{1/2}(C_1\boldsymbol{\psi}^1 + C_2\boldsymbol{\psi}^2 + A_1\boldsymbol{\xi}^1 + A_2\boldsymbol{\xi}^2)\chi + \mathbf{l}(\mathbf{x}) + \mathbf{w}, \quad \mathbf{w} \in V_{-\delta}^2(\omega) \tag{2.11}$$

$$l_1(\mathbf{x}) = a_1 + bx_2, \quad l_2(\mathbf{x}) = a_2 - bx_1$$

where C_1, C_2, a_1, a_2, b are constants, and the following estimate is valid

$$|||\mathbf{v}||| \equiv \sum_{i=1}^{2}(|C_i| + |a_i|) + |b| + \|\mathbf{w}, V_{-\delta}^2(\omega)\| \leqslant \tag{2.12}$$

$$C\Big(\sum_{i=1}^{2}|A_i| + |\alpha_i| + \|\mathbf{F}^*, \overset{\circ}{V}_{-\delta}(\omega)\| + \|\mathbf{Q}^*, V_{-\delta}^{1/2}(\gamma)\|\Big)$$

Here and henceforth, the letter C will denote different constants dependent just on the domain ω, the number δ, and the operator coefficients. We note that $\mathbf{l}(\mathbf{x})$ is the body displacement as a rigid whole, so that the operators $\mathbf{L}, \mathbf{L}^R, \mathbf{B}, \mathbf{B}^R$ vanish on $\mathbf{l}$

We consider the first step in the iteration process. Because of the smoothness of $\mathbf{F}$ the imbedding, $\mathbf{F} \in V_{-\delta}^\circ(\omega)$ holds for any $\delta < 1$. Because of (2.1), the difference $Q_i^\pm - g_i$ is of the order $o(r)$, and hence, belongs to $V_{-\delta}^{1/2}(\gamma)$ for any $\delta < 1$. Therefore the right sides in (2.7) have the form of (2.9)

$$A_1 = A_2 = 0, \quad \alpha_i = g_i(t), \quad Q_r^* = -Q_1 + \alpha_1\chi\cos\theta/2, \quad Q_\theta^* = -Q_2 + \alpha_2\chi\cos\theta/2$$

According to (2.11) and (2.12) for $\forall t \in [0, T]$, the solution $\mathbf{v}^1$ is representable in the form

$$\mathbf{v}^1 = r^{1/2}(C_1^1\boldsymbol{\psi}^1 + C_2^1\boldsymbol{\psi}^2)\chi + \mathbf{l}^1 + \mathbf{w}^1 \tag{2.13}$$

$$l_1^1 = a_1^1 + b^1x_2, \quad l_2^1 = a_2^1 - b^1x_1$$

and the following estimate is valid

$$|||\mathbf{v}^1(t,\cdot)||| \equiv \sum_{i=1}^{2}(|C_i^1(t)|+|a_i^1(t)|)+|b^1(t)|+$$

$$\|\mathbf{w}^1(t,\cdot),\ V_{-\delta}^2(\omega)\| \leqslant C\Big(\sum_{i=1}^{2}|g_i(t)|+\|\mathbf{F}(t,\cdot),\ \overset{\circ}{V}_{-\delta}(\omega)\|+$$

$$\|\mathbf{Q}^*(t,\cdot),\ V_{-\delta}^{1/2}(\gamma)\|\Big)$$

There follows from this estimate that

$$|||\mathbf{v}^1, t||| \leqslant CI(t), \quad I(t) \equiv \sum_{i=1}^{2}|g_i,\ R^1, t|+|\mathbf{F}, \overset{\circ}{V}_{-\delta}(\omega), t|+|\mathbf{Q}^*,\ V_{-\delta}^{1/2}(\gamma), t| \tag{2.14}$$

The finiteness of the quantity I introduced in (2.14) follows from the assumption about the piecewise continuity of the loads in time.

Before considering the second step in the iteration process, we present some auxiliary calculations.

We let $\mathbf{L}^\circ$, $L^{R\circ}$, B°, $B^{R\circ}$ denote the principal parts of the operators $\mathbf{L}$, $\mathbf{L}^R$, $\mathbf{B}$, $\mathbf{B}^R$ with coefficients frozen in $\mathbf{x}_B$. It can be verified that the relationships

$$\mathbf{L}^\circ\mathbf{v} = \left\|\begin{matrix} L_r^\circ\mathbf{v} \\ L_\theta^\circ\mathbf{v}\end{matrix}\right\| = r^{-3/2}G^\circ\left\|\begin{matrix}\varphi_r^{\cdot\cdot}-{}^3/_2\alpha\varphi_r+\beta_-\varphi_\theta^{\cdot} \\ 2\alpha\varphi_\theta^{\cdot\cdot}-{}^3/_4\varphi_\theta+\beta_+\varphi_r^{\cdot}\end{matrix}\right\| \tag{2.15}$$

$$\mathbf{L}^{R\circ}\mathbf{v} = \left\|\begin{matrix} L_r^{R\circ}\mathbf{v} \\ L_\theta^{R\circ}\mathbf{v}\end{matrix}\right\| = r^{-3/2}G^\circ\left\|\begin{matrix}N\varphi_r^{\cdot\cdot}-{}^3/_2\alpha N\varphi_r+\beta_-N\varphi_\theta^{\cdot} \\ 2\alpha N\varphi_\theta^{\cdot\cdot}-{}^3/_4 N\varphi_\theta+\beta_+N\varphi_r^{\cdot}\end{matrix}\right\|+$$

$$r^{-3/2}\frac{E_*}{6}\left\|\begin{matrix}{}^3/_2S\varphi_r+S\varphi_\theta^{\cdot} \\ -3S\varphi_r^{\cdot}-2S\varphi_\theta^{\cdot\cdot}\end{matrix}\right\|$$

$$\mathbf{B}^\circ\mathbf{v} = \left\|\begin{matrix} B_r^\circ\mathbf{v} \\ B_\theta^\circ\mathbf{v}\end{matrix}\right\| = -r^{-1/2}\cos\frac{\theta}{2}G^\circ\left\|\begin{matrix}\varphi_r^{\cdot}-{}^1/_2\varphi_\theta \\ \gamma\varphi_r+2\alpha\varphi_\theta^{\cdot}\end{matrix}\right\|, \quad \theta=0,\ 2\pi$$

$$\mathbf{B}^{R\circ}\mathbf{v} = \left\|\begin{matrix} B_r^{R\circ}\mathbf{v} \\ B_\theta^{R\circ}\mathbf{v}\end{matrix}\right\| = -r^{-1/2}\cos\frac{\theta}{2}G^\circ\left\|\begin{matrix}N\varphi_r^{\cdot}-{}^1/_2N\varphi_\theta \\ \gamma N\varphi_r+2\alpha N\varphi_\theta^{\cdot}\end{matrix}\right\|+$$

$$r^{-1/2}\cos\frac{\theta}{2}\frac{E_*}{6}\left\|\begin{matrix}0 \\ 3S\varphi_r+2S\varphi_\theta^{\cdot}\end{matrix}\right\|, \quad \theta=0,\ 2\pi$$

$$\alpha=\frac{1-\nu^\circ}{1-2\nu^\circ}, \quad \beta_\pm=\frac{1\pm 2k^\circ}{2(1-2\nu^\circ)}, \quad \gamma=\frac{2-\nu^\circ}{1-2\nu^\circ}$$

hold for functions of a special kind, namely $v_r=\varphi_r(t,\ \theta)r^{1/2}$, $v_\theta=\varphi_\theta(t,\ \theta)r^{1/2}$

The dot in (2.15) denotes differentiation with respect to θ, while N and S are integral operators given by the formulas

$$N\varphi(t,\theta)=\int_0^t R_1^\circ(t,\tau)\varphi(\tau,\theta)\,d\tau, \quad S\varphi(t,\theta)=\int_0^t Z(t,\tau)\varphi(\tau,\theta)\,d\tau \tag{2.16}$$

We note that the second expression in (2.15) for $\mathbf{L}^{R\circ}$ consists of two terms, the first of which can be obtained formally from the first expression in (2.15) for $\mathbf{L}^\circ$ by replacing the functions $\varphi_r, \varphi_\theta$ by the functions $N\varphi_r$ and $N\varphi_\theta$. Similarly for the last two representations in (2.15) for $\mathbf{B}^{R\circ}$ and $\mathbf{B}^\circ$. This circumstance facilitates the calculations.

Let $C_1(t)$, $C_2(t)$ be arbitrary functions of the time. Calculations utilizing (2.15) show that

$$\mathbf{L}^{R\circ}(r^{1/2}C_1\boldsymbol{\psi}^1)=r^{-3/2}MC_1\boldsymbol{\zeta}^1, \quad \mathbf{B}^{R\circ}(r^{1/2}C_1\boldsymbol{\psi}^1)=0 \tag{2.17}$$

$$\mathbf{L}^{R\circ}(r^{1/2}C_2\boldsymbol{\psi}^2)=r^{-3/2}MC_2\boldsymbol{\zeta}^2, \quad \mathbf{B}^{R\circ}(r^{1/2}C_2\boldsymbol{\psi}^2)=\left\|\begin{matrix}0 \\ 2r^{-1/2}MC_2\end{matrix}\right\|$$

$$\Big(M\varphi(t)=\frac{4G^\circ(t)}{1-2\nu^\circ(t)}[N\nu^\circ\varphi(t)-\nu^\circ(t)N\varphi(t)]+\frac{E_*(t)}{3}S\varphi(k^\circ-1)(t)\Big)$$

Writing $Ng\varphi(t)$ means that the operator N given by (2.16) is evaluated on the product of the functions g and φ.

The assertion, proved exactly in the same manner as in Lemma 1 in /5/, is valid.

Lemma. Let

$$\mathbf{w} \in L^\infty [0, T; V_{-\delta}^2 (\omega)], \ \varphi \in L^\infty (0, T, R^1).$$

Then for $\forall t \in [0, T]$ the estimates hold (α is the exponent from condition (1.6))

$$\| \mathbf{L}^R (\mathbf{w}) (t, \cdot), V_{-\delta}^\circ (\omega) \| \leqslant C \int_0^t (t-\tau)^{-\alpha} \| \mathbf{w} (\tau, \cdot), \ V_{-\delta}^2 (\omega) \| d\tau \tag{2.18}$$

$$\| \mathbf{B}^R (\mathbf{w}) (t, \cdot), V_{-\delta}^{1/2} (\gamma) \| \leqslant C \int_0^t (t-\tau)^{-\alpha} \| \mathbf{w} (\tau, \cdot), \ V_{-\delta}^2 (\omega) \| d\tau \tag{2.19}$$

$$| S\varphi (t) | \leqslant C \int_0^t (t-\tau)^{-\alpha} | \varphi (\tau) | d\tau, \quad | M\varphi (t) | \leqslant C \int_0^t (t-\tau)^{-\alpha} | \varphi(\tau) | d\tau \tag{2.20}$$

For the second step in the iteration process (2.7) and (2.8) it is necessary to evaluate $\mathbf{F}^1, \mathbf{Q}^1$ and to represent them in a form analogous to (2.9)

$$\mathbf{F}^1 = r^{-1/2} m (A_1{}^1 \zeta^1 + A_2{}^1 \zeta^2) \chi + \mathbf{F}^{*1}, \quad \mathbf{Q}^1 = \begin{Vmatrix} Q_r{}^1 \\ Q_\theta{}^1 \end{Vmatrix} = \begin{Vmatrix} Q_r{}^{*1} \\ Q_\theta{}^{*1} - 2A_2{}^1 r^{-1/2} \chi \end{Vmatrix} \tag{2.21}$$

Taking account of (2.13) and (2.17), we obtain from the definition of $\mathbf{F}^1$

$$\begin{gathered} \mathbf{F}^1 = - \mathbf{L}^R (r^{1/2} \Sigma^1 \chi + \mathbf{l}^1 + \mathbf{w}^1) = \\ - \mathbf{L}^R (r^{1/2} \Sigma^1 \chi) - \mathbf{L}^R (\mathbf{w}^1) + \mathbf{L}^{R^\circ} (r^{1/2} \Sigma^1) \chi - \\ \mathbf{L}^{R^\circ} (r^{1/2} \Sigma^1) \chi = - r^{-1/2} (M C_1{}^1 \zeta^1 + M C_2{}^1 \zeta^2) \chi + \mathbf{F}^{*1} \\ \mathbf{F}^{*1} = - \mathbf{L}^R (\mathbf{w}^1) + \mathbf{L}^{R^\circ} (r^{1/2} \Sigma^1) \chi - \mathbf{L}^R (r^{1/2} \Sigma^1 \chi), \ \Sigma^1 = C_1{}^1 \psi^1 + C_2{}^1 \psi^2 \end{gathered} \tag{2.22}$$

We set $A_i{}^1 \equiv -m^{-1} M C_i{}^1$. For $\delta < 1/2$ the following inequality is valid

$$\| \mathbf{F}^{*1} (t, \cdot), \ V_{-\delta}^\circ (\omega) \| < C \int_0^t (t-\tau)^{-\alpha} ||| \mathbf{v}^1 (\tau, \cdot) ||| d\tau \tag{2.23}$$

An estimate of the first term in the expression for $\mathbf{F}^{*1}$ results from (2.18). Because of the smoothness of the coefficients, the difference between the true rheological characteristics and those frozen at the crack tip is of the order of r, hence, the difference between the highest terms of the second two components is of the order $r^{-1/2}$, i.e., belongs to the space $V_{-\delta}^\circ$ for $\delta < +1/2$. Therefore, in compliance with the lemma, the difference $\mathbf{L}^{R^\circ} - \mathbf{L}^R$ in the expression for $\mathbf{F}^{*1}$ will also not exceed the right side of (2.23).

Taking account of (2.13) and (2.17), we obtain from the definition of $\mathbf{Q}^1$

$$\begin{gathered} \mathbf{Q}^1 = \mathbf{B}^R (r^{1/2} \Sigma^1 \chi + \mathbf{l}^1 + \mathbf{w}^1) = \mathbf{B}^R (r^{1/2} \Sigma^1 \chi) + \\ \mathbf{B}^R (\mathbf{w}^1) + \mathbf{B}^{R^\circ} (r^{1/2} \Sigma^1) \chi - \mathbf{B}^{R^\circ} (r^{1/2} \Sigma^1) \chi = \\ \begin{Vmatrix} 0 \\ 2r^{-1/2} M C_2{}^1 \chi \end{Vmatrix} + \mathbf{Q}^{*1} \\ \mathbf{Q}^{*1} = \mathbf{B}^R (\mathbf{w}^1) + \mathbf{B}^R (r^{1/2} \Sigma^1 \chi) - \mathbf{B}^{R^\circ} (r^{1/2} \Sigma^1) \chi \end{gathered} \tag{2.24}$$

The next inequality has exactly the same foundation as (2.23) (with satisfaction of (2.19) instead of (2.18))

$$\| \mathbf{Q}^{*1} (t, \cdot), \ V_{-\delta}^{1/2} (\gamma) \| \leqslant C \int_0^t (t-\tau)^{-\alpha} ||| \mathbf{v}^1 (\tau, \cdot) ||| d\tau \tag{2.25}$$

Thus, the right side of $\mathbf{F}^1, \mathbf{Q}^1$ are represented in the second step of the iteration process in the form of (2.9):

$$A_i = A_i{}^1 \equiv -m^{-1} M C_i{}^1, \ \alpha_i = 0, \ i = 1 \ 2, \mathbf{F}^* = \mathbf{F}^{*1}, \ \mathbf{Q}^* = \mathbf{Q}^{*1} \tag{2.26}$$

We conclude from (2.26) and (2.20) that

$$| A_i{}^1 (t) | \leqslant C \int_0^t (t-\tau)^{-\alpha} | C_i{}^1 (\tau) | d\tau, \quad i = 1, 2 \tag{2.27}$$

The representations for $\mathbf{v}^2$ and the estimates for $||| \mathbf{v}^2 (t, \cdot) |||$ follow from (2.11) and (2.12):

$$\mathbf{v}^2 = r^{1/2}(C_1{}^2\psi^1 + C_2{}^2\psi^2 + A_1{}^1\xi^1 + A_2{}^1\xi^2)\chi + \mathbf{l}^2 + \mathbf{w}^2 \tag{2.28}$$
$$l_1{}^2 = a_1{}^2 + b^2x_2, \quad l_2{}^2 = a_2{}^2 - b^2x_1$$

$$|||\mathbf{v}^2(t,\cdot)||| \equiv \sum_{i=1}^{2}(|C_i{}^2(t)| + |a_i{}^2(t)|) + |b^2(t)| + \tag{2.29}$$
$$\|\mathbf{w}^2(t,\cdot), V^2_{-\delta}(\omega)\| \leqslant C\,(\sum_{i=1}^{2}|A_i{}^1(t)| + \|\mathbf{F}^{*1}(t,\cdot), V^0_{-\delta}(\omega)\| +$$
$$\|\mathbf{Q}^{*1}(t,\cdot), V^{1/2}_{-\delta}(\gamma)\|)$$

Collecting the estimates (2.23), (2.25), (2.27) and (2.29) together, we obtain the inequality

$$|||\mathbf{v}^2(t,\cdot)||| \leqslant C\int_0^t (t-\tau)^{-\alpha}|||\mathbf{v}^1(\tau,\cdot)|||\,d\tau \tag{2.30}$$

We now consider the third step in the iteration process. Compared with (2.13), the formula (2.28) contains two terms of a new kind, namely, the vectors ξ^i, $i = 1, 2$ are present in the principal part. Calculations utilizing (2.15) show that

$$\mathbf{L}^{R^\circ}(r^{1/2}A_1\xi^1) = r^{-1/2}DA_1\zeta^1, \mathbf{B}^{R^\circ}(r^{1/2}A_1\xi^1) = 0 \tag{2.31}$$
$$\mathbf{L}^{R^\circ}(r^{1/2}A_2\xi^2) = r^{-1/2}DA_2\zeta^2, \quad \mathbf{B}^{R^\circ}(r^{1/2}A_2\xi^2) = \begin{Vmatrix} 0 \\ 2r^{-1/2}DA_2{}^1 \end{Vmatrix}$$

Here D is an integral operator of the form

$$D\varphi(t) = -mN\varphi(t) + \frac{E_*(t)}{6}S\varphi(t) \tag{2.32}$$

Further calculations are a literal duplication of the process indicated in the second step with the difference that in the representations of the right sides of $\mathbf{F}^2, \mathbf{Q}^2$, analogous to (2.21), the coefficients $A_i{}^2$ are expressed by the formula

$$A_i{}^2 = -m^{-1}(MC_i{}^2 - DA_i{}^1), \quad i = 1, 2 \tag{2.33}$$

The equality (2.33) generalizes (2.26); in addition to (2.17), also (2.31) was utilized in its derivation.

We consequently obtain a representation for $\mathbf{v}^3$ and an estimate for $|||\mathbf{v}^3(t,\cdot)|||$ in terms of $|||\mathbf{v}^2(t,\cdot)|||$:

$$\mathbf{v}^3 = r^{1/2}(C_1{}^3\psi^1 + C_2{}^3\psi^2 + A_1{}^2\xi^1 + A_2{}^2\xi^2)\chi + \mathbf{l}^3 + \mathbf{w}^3 \tag{2.34}$$

$$|||\mathbf{v}^3(t,\cdot)||| \leqslant C\int_0^t (t-\tau)^{-\alpha}|||\mathbf{v}^2(\tau,\cdot)|||\,d\tau \tag{2.35}$$

The next steps in the process are considered exactly as was the third step, hence, the following assertions are valid, in complete analogy to (2.34), (2.33) and (2.35): the solution $\mathbf{v}^n$ is representable in the form

$$\mathbf{v}^n = r^{1/2}(C_1{}^n\psi^1 + C_2{}^n\psi^2 + A_1^{n-1}\xi^1 + A_2^{n-1}\xi^2)\chi + \mathbf{l}^n + \mathbf{w}^n, n = 3, 4, \ldots$$
$$l_1{}^n = a_1{}^n + b^nx_2, \quad l_2{}^n = a_2{}^n - b^nx_1$$

where the coefficients $A_i{}^{n-1}$ are defined by the recursion formulas

$$A_i{}^{n-1} = -m^{-1}(MC_i{}^{n-1} + DA_i{}^{n-2}), \quad i = 1, \; 2, \quad n = 3, \; 4, \ldots \tag{2.36}$$

and the following estimate holds

$$|||\mathbf{v}^n(t,\cdot)||| = \sum_{i=1}^{2}(|C_i{}^n(t)| + |a_i{}^n(t)|) + |b^n(t)| + \|\mathbf{w}^n(t,\cdot), V^2_{-\delta}(\omega)\| \leqslant \tag{2.37}$$
$$C\int_0^t (t-\tau)^{-\alpha}|||\mathbf{v}^{n-1}(\tau,\cdot)|||\,d\tau, \quad n = 3, 4, \ldots$$

The inequalities (2.30) and (2.37) are the fundamental relationships for the proof of the theorem.

We conclude from (2.30) and (2.14) that

$$|||\mathbf{v}^2(t,\cdot)||| \leqslant C\int_0^t (t-\tau)^{-\alpha}\, d\tau I(T) = \frac{Ct^\beta}{\beta} I(T), \quad \beta \equiv 1-\alpha \tag{2.38}$$

We introduce the integral

$$I_n(t) = \int_0^t \tau^{n\beta}(t-\tau)^{-\alpha}\, d\tau = D_n t^{(n+1)\beta}, \quad D_n \equiv \frac{\Gamma(\beta)\Gamma(n\beta+1)}{\Gamma[(n+1)\beta+1]} \tag{2.39}$$

Substituting (2.38) into (2.37) for $n=3$ and utilizing the definitions (2.39), we find

$$|||\mathbf{v}^3(t,\cdot)||| \leqslant \frac{C}{\beta} I(T)\int_0^t \tau^\beta (t-\tau)^{-\alpha}\, d\tau = \frac{C}{\beta} D_1 t^{2\beta} I(T)$$

Applying (2.37) successively for $n = 4, 5, \ldots$, we arrive at the estimate

$$|||\mathbf{v}^n(t,\cdot)||| \leqslant \frac{C}{\beta} D_1 D_2 \ldots D_{n-2} I(T) = \frac{C\Gamma(\beta+1)[C\Gamma(\beta)]^{n-2}}{\beta\Gamma[(n-1)\beta+1]} t^{(n-1)\beta} I(T) \tag{2.40}$$

According to Lemma 5 in /5/, a series with the common term equal to the right side in (2.40) will converge for any t. The convergence of the series with the common terms C_i^n, a_i^n, b^n, A_i^n, $\mathbf{w}^n$, hence follows, which proves the representation (2.2), where

$$C_1(t) = \sum_{n=1}^\infty C_1^n(t), \quad A_i(t) = \sum_{n=1}^\infty A_i^n(t), \quad a_i(t) = \sum_{n=1}^\infty a_i^n(t), \; i = 1, 2 \tag{2.41}$$

$$b(t) = \sum_{n=1}^\infty b^n(t), \quad \mathbf{o}(r) = \sum_{n=1}^\infty \mathbf{w}^n(t)$$

To prove (2.4), we sum (2.36) over $n = 3, 4, \ldots$ and we append (2.26), and taking account of (2.41) we obtain the equation

$$A_i = -m^{-1}(MC_i + DA_i), \; i = 1, 2 \tag{2.42}$$

Utilizing the definition of the operators (2.16) and (2.32), and the notation for m in (2.9), it can be confirmed that (2.42) and (2.4) are in agreement.

Formulas (2.3) are obtained by direct substitution of (2.2) into (1.1) and (1.3). The theorem is proved.

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Translated by M.D.F.